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THE METHODS OF RITZ, GALERKIN, AND COMPLEMENTARY ENERGY APPLIED TO SOME NONCONSERVATIVE PROBLEMS OF ELASTIC STABILITY

JULY 1972

BENÉT WEAPONS LABORATORY

WATERVLIET ARSENAL
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BY

G.L. ANDERSON

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BENÉT WEAPONS LABORATORY

WATERVLIET ARSENAL

Watervliet, New York

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THE METHODS OF RITZ, GALERKIN, AND COMPLEMENTARY ENERGY APPLIED TO SOME NONCONSERVATIVE PROBLEMS OF ELASTIC STABILITY

ABSTRACT

The well-known problem of Beck and two problems due to Hauger which arise in the theory of nonconservative elastic stability are reconsidered here. Approximate values of the respective critical loads are computed by means of the methods of Ritz, Galerkin, and complementary energy in conjunction with a formal variational expression and the adjoint variational principle. For these examples, it was found that the speed of convergence of the procedures is somewhat better in the case of the method of complementary energy, but the Galerkin procedure is probably the simplest to apply. The numerical calculations demonstrate that the previously published values of the critical load intensity for Hauger's two problems are about 5% and 28% too high.

Cross-Reference Data

Stability

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INTRODUCTION

In conservative vibration and static buckling problems, a variety of methods of approximation with their theoretical foundations residing in the calculus of variations have been applied to determine numerical values of natural frequencies of vibration and buckling loads. The familiar methods of Ritz and Galerkin are probably the most prominent, and the theoretical basis for the application of these methods to self-adjoint boundary value problems is rather well established. The principle of stationary complementary energy has also been used for such problems, and despite the accuracy of the method, which is particularly advantageous in computing the lowest as well as higher eigenvalues, the technique has not received widespread attention, possibly because of the extra effort required in computing expressions for stresses (bending moments in beam and plate problems) once an apposite set of coordinate functions has been selected.

For nonconservative problems of elastic stability, in which nonself-adjoint boundary value problems arise, the question regarding convergence of the Ritz and Galerkin procedures has been answered in the affirmative only for special, albeit fairly broad classes of problems. Indeed, the method of complementary energy has received no attention at all. This seems rather unfortunate since in nonconservative stability problems an accurate knowledge of the variation of frequency in at least the first two modes with the load parameter is essential for the effective determination of the value of the critical

load. In this report we shall apply to three nonself-adjoint boundary value problems these various techniques in conjunction with a formal variational expression and the so-called adjoint variational principle, and their relative merits will be compared and contrasted.

BECK'S PROBLEM

Let us consider perhaps the best known problem of the theory of stability of non-conservatively loaded structures, namely, Beck's problem [1]. This problem consists of determining the smallest load P that is applied to the free end of a cantilever beam and which remains tangent to the deformed axis of the beam throughout the process of deformation, such that the amplitude of the resulting oscillations of the beam increases exponentially with time. Loss of stability in this way is often called flutter. The differential equation and boundary conditions of Beck's problem are

$$EI \frac{\partial^4 w}{\partial \bar{x}^4} + P \frac{\partial^2 w}{\partial \bar{x}^2} + \rho A \frac{\partial^2 w}{\partial \bar{t}^2} = 0, \qquad o \leq \bar{x} \leq \ell, \tag{1}$$

$$w(o,\bar{t}) = \frac{\partial w}{\partial \bar{x}}(o,\bar{t}) = \frac{\partial^2 w}{\partial \bar{x}^2}(\ell,\bar{t}) = \frac{\partial^3 w}{\partial \bar{x}^3}(\ell,\bar{t}) = 0, \tag{2}$$

where $w(\bar{x},\bar{t})$ is the transverse deflection of the beam at the point $\bar{x}$ $(o \leq \bar{x} \leq \ell)$ and at time $\bar{t}$ $(\bar{t} > o)$, EI is the flexural rigidity, ρ the density, A the area of the cross section, and ℓ the length of the beam.

It is convenient to introduce dimensionless time and space coordinates. Thus, we set

$$\bar{x} = \ell x, \qquad \bar{t} = ct, \qquad c^2 = \frac{\rho A \ell^4}{EI}, \qquad Q = \frac{P\ell^2}{EI}$$

so that (1) and (2) may be written as

$$\frac{\partial^4 w}{\partial x^4} + Q\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial t^2} = 0, \qquad o<x<1, \tag{3}$$

$$w(o,t) = w'(o,t) = w''(1,t) = w'''(1,t) = 0, \tag{4}$$

where the prime symbol ' denotes differentiation with respect to x.

Next we assume that

$$w(x,t) = y(x)e^{i\omega t}, \tag{5}$$

where ω is the natural frequency parameter. Substitution of (5) into (3) and (4) yields the following nonself-adjoint boundary value problem:

$$y^{IV}(x) + Q\,y''(x) - \omega^2 y(x) = 0, \qquad o<x<1, \tag{6}$$

$$y(o) = y'(o) = y''(1) = y'''(1) = 0. \tag{7}$$

It is our objective to determine from (6) and (7) the variation of the frequency parameter ω as a function of the load parameter Q. Since the eigenvalues, i.e., the natural frequencies, of nonself-adjoint boundary value problems may be complex numbers (recall that self-adjoint boundary value problems can have only real eigenvalues), the phenomenon of flutter can occur. This means that if the imaginary part of ω is negative, then the amplitude of the oscillation will grow exponentially

in time. Generally, in those problems in which the kinetic method must be applied, the frequencies of the various modes are real for sufficiently small values of Q and vary as the magnitude of the load increases, the frequency of the first mode eventually increasing and that of the second mode decreasing until the two coalesce at a value of Q denoted by Q_{cr}. For $Q<Q_{cr}$, ω is a real number, but, for $Q>Q_{cr}$, ω is a complex number and flutter occurs. Beck [1] solved (6) and (7) exactly and found that the value of the critical load, Q_{cr}, is

$$Q_{cr} = 2.031\pi^2. \tag{8}$$

It shall be our intention here to apply several methods of approximation to Beck's problem in order to gain some understanding regarding the accuracy of the approximations, having available the exact value in (8) to serve as the basis for comparison. Ultimately, we will apply those methods which appear to be more accurate or most convenient to two problems for which the exact values of the critical loads are not known.

THE RITZ METHOD

The methods of approximation are to be based upon certain variational formulations of the fundamental problem in (6) and (7). Our first approach will parallel variational formulations used by Leipholz [2] and Mote [3]. We multiply (6) by $\delta y(x)$ and form the integral of the result:

$$\int_0^1 [y^{IV}(x) + Q\, y''(x) - \omega^2 y(x)]\delta y(x)dx = 0.$$

Integrating by parts and using the boundary conditions (7), we obtain

$$\delta \frac{1}{2} \int_{o}^{1} [(y'')^2 - Q(y')^2 - \omega^2 y^2]dx + Qy'(1)\delta y(1) = 0. \tag{9}$$

The nonconservative nature of the load is manifested in the presence of the variational term $Qy'(1)\ \delta y(1)$ in (9). A complete functional in the usual sense does not exist here because the nonconservative load Q does not possess a potential. Relations of the type shown in (9) are often called "variational principles." This usage is not consistent with that of the classical calculus of variations because there is no functional which is stationary.

Leipholz [4] - [8] has demonstrated that Galerkin's procedure in various forms can be applied to stability problems of the type under consideration here. We shall assume that

$$y(x) = \tilde{y}(x) = \sum_{n=1}^{N} a_n\, y_n(x), \tag{10}$$

where the a_n's are unknown constants and the coordinate functions $y_n(x)$ are selected so as to satisfy all the boundary conditions in (7). In particular, suppose that we choose

$$y_n(x) = A_n x^{n+1} + B_n x^{n+2} + C_n x^{n+3}, \quad n \geq 1, \tag{11}$$

which obviously satisfy the geometric boundary conditions $y_n(o) = y_n'(o) = 0$. The constants A_n, B_n, C_n will now be chosen such that the natural boundary conditions

$$y_n''(1) = y_n'''(1) = 0 \tag{12}$$

are satisfied.

Substitution of (11) into (12) yields

$$n(n+1)A_n + (n+1)(n+2)B_n = -(n+2)(n+3)C_n$$

$$(n-1)nA_n + n(n+2)B_n = -(n+2)(n+3)C_n,$$

from which we find

$$A_n = \frac{(n+3)(n+2)}{n(n+1)} C_n, \qquad B_n = -2\frac{(n+3)}{(n+1)} C_n.$$

It is convenient to set $C_n = n(n+1)/2$, so that (11) becomes

$$y_n(x) = \frac{1}{2}[(n+2)(n+3)x^{n+1} - 2n(n+3)x^{n+2} + n(n+1)x^{n+3}], \quad n \geq 1. \tag{13}$$

Let us now consider (9) and the sequence of steps followed formally in the application of the Ritz method. If we substitute (10) into (9), we find

$$\sum_{n=1}^{N}\sum_{k=1}^{N} a_n\{\int_0^1 [y_n'' y_k'' - Q y_n' y_k' - \omega^2 y_n y_k]dx + Q y_n'(1)y_k(1)\}\delta a_k = 0,$$

but since the δa_k are arbitrary we arrive at

$$\sum_{n=1}^{N} a_n[A_{nk} + Q B_{nk} - \omega^2 C_{nk}] = 0, \tag{14}$$

where

$$A_{nk} = \int_0^1 y_n''(x)y_k''(x)dx, \quad B_{nk} = y_n'(1)y_k(1) - \int_0^1 y_n'(x)y_k'(x)dx,$$

$$C_{nk} = \int_0^1 y_n(x)y_k(x)dx.$$

Now (14) represents a system of N algebraic equations in the N unknowns, a_n, and the condition for the existence of a nontrivial solution is that

$$\det(A_{nk} + Q B_{nk} - \omega^2 C_{nk}) = 0. \tag{15}$$

Eq. (15) was solved numerically for N = 2. The value of the critical load parameter was found to be

$$Q_{cr} = 2.193\pi^2, \tag{16}$$

which is to be compared with the exact value in (8).

THE GALERKIN PROCEDURE

We may carry the analysis forward a little more to show that for (9) and (13) the Ritz and Galerkin methods are identical. In view of the fact that the $y_n(x)$'s given in (13) satisfy the boundary conditions (7), we may integrate by parts to obtain

$$A_{nk} = \int_0^1 y_n''(x)y_k''(x)dx = \int_0^1 y_n \;(x)y_k(x)dx,$$

$$B_{nk} = y_n'(1)y_k(1) - \int_0^1 y_n'(x)y_k'(x)dx = \int_0^1 y_n''(x)y_k(x)dx.$$

Substitution of these last two results into (14) leads to

$$\sum_{n=1}^{N} a_n \int_0^1 [y_n^{IV}(x) + Q\, y_n''(x) - \omega^2 y_n(x)]y_k(x)dx = 0,$$

or, by virtue of (10),

$$\int_0^1 [\tilde{y}^{IV}(x) + Q\, \tilde{y}''(x) - \omega^2\tilde{y}(x)]y_k(x)dx = 0, \tag{17}$$

which we recognize as the method of Galerkin. Eq. (17) leads to precisely the same numerical result as did the Ritz approach as embodied in (14). In practice it is generally easier from the point of view of the numerical work involved to deal with (17) rather than (9).

LEVINSON'S METHOD

Levinson [9] obtained an approximate solution for Beck's problem by modifying the form of (9) and then making the numerical calculation. We shall record here only the outline of his technique.

The identity

$$y'(1)\delta y(1) = \delta[y'(1)y(1)] - y(1)\delta y'(1)$$

is inserted into (9), the result being

$$\delta\{\frac{1}{2}\int_0^1[(y'')^2 - Q(y')^2 - \omega^2y^2]dx + Qy'(1)y(1)\} - Qy(1)\delta y'(1)=0.$$

It was subsequently argued that this last equation is equivalent to

$$\delta\{\frac{1}{2}\int_0^1[(y'')^2 - Q(y')^2 - \omega^2y^2]dx + Qy'(1)y(1) = 0 \tag{18}$$

with the auxiliary condition

$$\delta y'(1) = 0. \tag{19}$$

Let us assume that y(x) is again approximated by (10) and (13). Substitution of (10) into (18) leads to

$$\delta\{\frac{1}{2}\sum_{n=1}^{N}\sum_{k=1}^{N} a_n a_k \int_0^1[y_n''y_k'' - Qy_n'y_k' - \omega^2 y_n y_k]dx +$$

$$+ Q\sum_{n=1}^{N}\sum_{k=1}^{N} a_n a_k y_n'(1)y_k(1)\} = 0$$

or

$$\sum_{n=1}^{N}\sum_{k=1}^{N} a_n \int_0^1[y_n''y_k'' - Qy_n'y_k' - \omega^2 y_n y_k]dx\,\delta a_k +$$

$$+ Q\sum_{n=1}^{N}\sum_{k=1}^{N} a_n y_n'(1)y_k(1)\delta a_k + Q\sum_{n=1}^{N}\delta a_n y_n'(1)\sum_{k=1}^{N} a_k y_k(1) = 0. \tag{20}$$

But the auxiliary condition (19) leads to

$$\sum_{n=1}^{N} \delta a_n \, y_n'(1) = 0,$$

so (20) reduces to

$$\sum_{n=1}^{N} \sum_{k-1}^{N} a_n \{\int_0^1 y_n'' y_k'' dx + Q[y_n'(1) y_k(1) - \int_0^1 y_n' y_k' dx] -$$

$$-\omega^2 \int_0^1 y_n y_k dx\} \delta a_k = 0,$$

which is nothing more than (14).

Therefore, Levinson's method leads to exactly the same frequency equation as do the Ritz and Galerkin procedures. In general, it would seem that the Galerkin approach would be the easiest to apply, provided that the admissible functions satisfy all the geometric and natural boundary conditions of the given problem. Leipholz [2] has discussed how the Galerkin procedure must be modified in the event that some of the geometric or natural boundary conditions are violated by the form of the selected coordinate functions $y_n(x)$.

THE METHOD OF COMPLEMENTARY ENERGY

Washizu [10] - [12] has applied the principle of stationary complementary energy to some conservative vibration and stability problems. This technique was also discussed by Reissner [13], Libby and Sauer [14], and van de Vooren and Greidanus [15]. When applied to conservative problems, the method based on the principle of stationary complementary energy, according to [12] and [15], is really identical to Grammel's method [16], which, in theory, involves

an integral equation formulation of the eigenvalue problem and the Green's function associated with the differential operator and boundary conditions of the original problem. Leipholz [17] - [18] has discussed the convergence of Grammel's procedure and has presented an extended version of it. Using a two-term approximation in certain free vibration problems associated with the flexural deformation of a beam, Washizu showed that the principle of stationary complementary energy led to an excellent approximate value for the lowest frequency and an approximate value for the frequency of the second mode that represented a substantial improvement relative to the corresponding value obtained by means of the familiar Ritz procedure. Consequently, since accurate values of both the first and second modes are required for the determination of the critical load parameter in those nonconservative stability problems in which the kinetic method must be used, it seems reasonable to speculate that the method described by Washizu could be a useful tool for obtaining an approximate solution of such problems.

In order to bring into play the concept of complementary energy, we set

$$M(x) = -y''(x),$$

where $M(x)$ is related to the bending moment, so that (9) becomes

$$\delta\frac{1}{2}\int_0^1 [M^2 - Q(y')^2 - \omega^2 y^2]dx + Qy'(1)\delta y(1) = 0, \tag{21}$$

under the subsidiary conditions

$$M''(x) - Qy''(x) + \omega^2 y(x) = 0, \qquad o<x<1, \tag{22}$$

$$y(o) = y'(o) = M(1) = M'(1) = 0. \tag{23}$$

Note that (22) and (23) are equivalent to (6) and (7). In (21), we shall consider the variation of both $M(x)$ and $y(x)$.

For $y(x)$ we once again intend to use (10) and (13), and the expression for $M(x)$ shall be obtained by integrating (22) subject to the last two conditions in (23). Now substitution of (10) into (22) leads to

$$M''(x) = \sum_{n=1}^{N} a_n M_n''(x), \tag{24}$$

where we define

$$M_n''(x) = Qy_n''(x) - \omega^2 y_n(x), \quad M_n(1) = M_n'(1) = 0. \tag{25}$$

In view of $M(1) = M'(1) = 0$, two integrations of (24) yield

$$M(x) = \sum_{n=1}^{N} a_n M_n(x), \tag{26}$$

whereas two integrations of (25) lead to

$$\begin{aligned} M_n(x) &= Q[y_n(x) + y_n'(1)(1-x) - y_n(1)] - \omega^2 \int_1^x (x-t) y_n(t) dt \\ &= Q\, f_n(x) - \omega^2 g_n(x), \end{aligned} \tag{27}$$

where

$$f_n(x) = n-(n+3)x + \frac{x^{n+1}}{2} [(n+2)(n+3)-2n(n+3)x + n(n+1)x^2],$$

$$g_n(x) = \frac{2(3n+10)}{(n+4)(n+5)} - \frac{6x}{n+4} + \frac{x^{n+3}}{2}\left(1 - \frac{2nx}{n+4} + \frac{n(n+1)x^2}{(n+4)(n+5)}\right).$$

Upon inserting (10) and (26) into (21), we obtain

$$\delta\frac{1}{2} \sum_{n=1}^{N} \sum_{k=1}^{N} a_n a_k \int_0^1 [M_n(x)M_k(x) - Qy_n'(x)y_k'(x) - \omega^2 y_n(x)y_k(x)]dx +$$

$$+ Q \sum_{n=1}^{N} \sum_{k=1}^{N} a_n \delta a_k y_n'(1) y_k(1) = 0.$$

Performing the indicated variation in this last result, we find

$$\sum_{n=1}^{N} a_n \{ \int_0^1 [M_n(x)M_k(x) - Qy_n'(x)y_k'(x) - \omega^2 y_n(x)y_k(x)]dx +$$

$$+ Qy_n'(1)y_k(1)\} = 0.$$

The numerical solution of the above equation for N = 2 leads to

$$Q_{cr} = 2.095\pi^2 \tag{28}$$

for the critical load in Beck's problem, With respect to the Ritz and Galerkin methods, the complementary energy method trades increased complexity for possibly better results.

THE METHOD OF THE ADJOINT VARIATIONAL PRINCIPLE

In the preceding paragraphs we applied the methods of Ritz, Galerkin, and complementary energy to a nonself-adjoint boundary value problem by means of the variational expression given in (9). The numerical results for the value of the critical load obtained in this fashion with a two-term approximation were slightly greater than the exact value given in (8), the best approximation being given by the method of complementary energy.

Since the applied load in Beck's problem is nonconservative, it does not possess a potential, and thus no functional, viz. no Lagrangian function, exists. Consequently, a complete variational formulation in the classical sense cannot be stated for this problem. Incomplete variational formulations of the type appearing in (9), while they often

provide useful numerical schemes, as we have seen in our calculations above, leave something to be desired in the eyes of the mathematical purist who prefers a sense of completeness in the theory underlying his numerical operations. In this regard, the papers by Finlayson and Scriven [19] and [20] are well worth reading. Since no functional exists for Beck's problem, we are not finding a stationary value in (9). Eq. (9) serves merely as a systematic means for acquiring approximate values for the critical load. As we demonstrated above, (9) is entirely equivalent to Galerkin's method, which is perhaps the best known example of a class of techniques known as the method of weighted residuals.

In the following paragraphs we shall introduce a boundary value problem which is the adjoint of the original boundary value problem. In terms of the original and adjoint problems, we will then be able to state a functional, the stationary value of which, in the sense of the calculus of variations, yields the differential equations and boundary conditions for both problems. In [19] it was stated that there is some evidence that this so-called adjoint variational principle leads to slightly better results than does the Galerkin method and that the speed of convergence might also be better. As is perhaps to be expected, the somewhat more accurate results obtained with the adjoint variational principle are acquired at the expense of increased complexity and labor.

The importance of the adjoint problem in nonself-adjoint eigenvalue problems was discussed by Roberts [21], Ballio [22] - [23]

applied the concept of the adjoint problem to a couple of nonconservative stability problems, including Beck's problem. Nemat-Nasser and Herrmann [24] also determined the boundary value problem adjoint to Beck's problem, and Prasad and Herrmann [25] described the usefulness of the adjoint problem in solving nonconservative stability problems. The method described in [25] has been generalized by Anderson and Walter [26] and Anderson [27] to include the effects of internal and external damping in nonconservative stability problems.

It was shown in [24] - [26] that the eigenvalue problem that is adjoint to (6) and (7) is

$$v^{IV}(x) + Q\,v''(x) - \omega^2 v(x) = 0, \qquad o<x<1, \tag{29}$$

$$v(o) = v'(o) = v''(1) + Q\,v(1) = v'''(1) + Q\,v'(1) = 0, \tag{30}$$

which is often called Peut's problem. Moreover, it was shown in [25], [26], and [28] that both the original problem (6), (7) and the adjoint problem (29), (30) can be generated from the variational principle

$$\delta\{\int_o^1 [y''v'' - Qy'v' - \omega^2 yv]dx + Qy'(1)v(1)\} = 0, \tag{31}$$

where now we are dealing with the problem of finding a stationary value for the given functional. We shall now use (31) in place of (9) for the purpose of determining approximate values for the critical load in Beck's problem. It is essential to keep in mind that the eigenvalues of the adjoint problem are identical to the eigenvalues of the original problem.

To obtain a suitable set of coordinate functions for the adjoint problem, let us assume that

$$v(x) = \tilde{v}(x) = \sum_{n=1}^{N} b_n v_n(x). \tag{32}$$

Moreover, since the geometric boundary conditions in (30) are the same as those in (7), we shall assume that the $v_n(x)$'s are of the form shown in (11), i.e.,

$$v_n(x) = A_n x^{n+1} + B_n x^{n+2} + C_n x^{n+3}, \qquad n \geq 1. \tag{33}$$

If we now require (33) to satisfy the two boundary conditions at $x = 1$ that are stated in (30), we can show, following the method applied above to (11) and (12), that the coefficients A_n, B_n, C_n can be selected such that (33) assumes the form

$$v_n(x) = \alpha_n x^{n+1} - 2\beta_n x^{n+2} + \gamma_n x^{n+3}, \tag{34}$$

where

$$\alpha_n = Q^2 + 2n(n+2)\, Q + (n+1)(n+2)^2(n+3),$$

$$\beta_n = Q^2 + 2n(n+1)\, Q + n(n+1)(n+2)(n+3),$$

$$\gamma_n = Q^2 + 2(n^2-1)\, Q + n(n+1)^2(n+2).$$

If we now substitute (10) and (32) into (31), we obtain

$$\delta \sum_{n=1}^{N} \sum_{k=1}^{N} a_n b_k I_{nk} = 0, \tag{35}$$

where

$$I_{nk} = \int_0^1 [y_n''(x)v_k''(x) - Q\, y_n'(x)v_k'(x) - \omega^2 y_n(x)v_k(x)]dx + Q\, y_n'(1)v_k(1). \tag{36}$$

Performing the variation in (35), we find

$$\sum_{n=1}^{N} \sum_{k=1}^{N} (a_n\, I_{nk}\, \delta b_k + b_n\, I_{kn}\, \delta a_k) = 0,$$

from which, in the usual way, it follows that

$$\sum_{n=1}^{N} a_n\, I_{nk} = 0, \qquad \sum_{n=1}^{N} b_n\, I_{kn} = 0. \tag{37}$$

The frequency equations are then

$$\det (I_{nk}) = 0 \quad \text{and} \quad \det (I_{kn}) = 0. \tag{38}$$

But these equations are equivalent because the value of a determinant remains unchanged upon interchanging its rows and columns. This is simply a manifestation of the fact that the eigenvalues of the original and adjoint problems are identical.

Using the coordinate functions in (13) and (34), which satisfy all the geometric and natural boundary conditions in (7) and (30), we may use either determinant in (38) to evaluate the critical load. The value reported in [25] and [28] is for $N = 2$

$$Q_{cr} = 1.971\pi^2, \tag{39}$$

somewhere lower than the exact value given in (8).

Let us go one step further without consideration of (36) and (38). Recalling that in the application of the Ritz method with a functional of the type given in (31), we are not, in general, required to select

coordinate functions which satisfy both the geometric and natural boundary conditions of the given eigenvalue problem. Indeed, we are really compelled to satisfy only the geometric boundary conditions, although this usually means that the speed of convergence of the method is slowed somewhat depending upon the nature of the problem under consideration. Let us now assume that the functions $y_n(x)$ and $v_k(x)$ in (36) satisfy only the geometric boundary conditions

$$y_n(o) = y_n'(o) = v_n(o) = v_n'(o) = 0. \tag{40}$$

In particular, we are assuming now that $y_n(x)$ and $v_n(x)$ are not necessarily the functions defined above in (13) and (34).

Integrating by parts and using (40), we can easily demonstrate that

$$\int_o^1 y_n''(x)v_k''(x)dx = \int_o^1 y_n\ (x)v_k(x)dx + y_n''(1)v_k'(1) - y_n'''(1)v_k(1),$$

$$\int_o^1 y_n'(x)v_k'(x)dx = -\int_o^1 y_n''(x)v_k(x)dx + y_n'(1)v_k(1),$$

so that (36) becomes

$$I_{nk} = \int_o^1 [y_n\ (x) + Q\,y_n''(x) - \omega^2 y_n(x)]v_k(x)dx + y_n''(1)v_k'(1) - y_n'''(1)v_k(1).$$

Therefore, the first relationship in (34) can be written as

$$\sum_{n=1}^{N} a_n\{\int_o^1 [y_n\ (x) + Q\,y_n''(x) - \omega^2 y_n(x)]v_k(x)dx + y_n''(1)v_k'(1) - y_n'''(1)v_k(1)\} = 0$$

or, equivalently

$$\int_0^1 [\tilde{y}^{IV}(x) + Q\,\tilde{y}''(x) - \omega^2\tilde{y}(x)]v_k(x)dx +$$

$$+ \tilde{y}''(1)v_k'(1) - \tilde{y}'''(1)v_k(1) = 0. \tag{41}$$

Eq. (41) bears a very strong resemblance to the so-called semi-extended Galerkin equations for nonself-adjoint eigenvalue problems as discussed by Leipholz [2]. Note that here, however, the weight functions are the coordinate functions of the adjoint problem $v_k(x)$ rather than the coordinate functions of the original eigenvalue problem. The presence of the terms $\tilde{y}''(1)v_k'(1) - \tilde{y}'''(1)v_k(1)$ in (41) serves to compensate for the fact that the coordinate functions $y_n(x)$ do not satisfy the natural boundary conditions $y''(1) = y'''(1) = 0$ of the original eigenvalue problem. Hence, (41) may be considered a generalization of the results presented in [2].

If the coordinate functions $y_n(x)$ satisfy the natural boundary conditions of the original problem, then (41) reduces to

$$\int_0^1 [\tilde{y}^{IV}(x) + Q\,\tilde{y}''(x) - \omega^2\tilde{y}(x)]v_k(x)dx = 0, \tag{42}$$

which is analogous to the classical form of the Galerkin equations. To this point we have assumed that the weight functions $v_k(x)$ satisfy only the geometric boundary conditions of the adjoint problem. The question of whether the procedure embodied in (42) converges requires further study. If this procedure does converge, it is very probable that it will converge even more rapidly if the $v_k(x)$'s are also required to satisfy the natural boundary conditions of the adjoint problem.

If $y_n(x)$ and $v_n(x)$ are defined by (13) and (34), then (42) is equivalent to (37). In general, the numerical work is less complicated in (42) then in (37) with I_{nk} given by (36).

THE COMPLEMENTARY ENERGY AND ADJOINT VARIATIONAL METHODS

Now let

$$M(x) = - y''(x), \qquad N(x) = - v''(x) \tag{43}$$

so that (31) can be expressed as

$$\delta\{\int_0^1 [M(x)N(x) - Q\, y'(x)v'(x) - \omega^2 y(x)v(x)]dx + Q\, y'(1)v(1)\} = 0. \tag{44}$$

The "bending moments" M(x) and N(x) are subject to the auxiliary conditions

$$\begin{aligned} M''(x) &= Q\, y''(x) - \omega^2 y(x), \\ M(1) &= M'(1) = 0, \end{aligned} \tag{45}$$

$$\begin{aligned} N''(x) &= Q\, v''(x) - \omega^2 v(x), \\ N(1) &= Q\, v(1), \quad N'(1) = Q\, v'(1). \end{aligned} \tag{46}$$

Because we have introduced the adjoint problem and consequently a variational principle, we may view (44) as a replacement for (21) with (45) and (46) being introduced in analogy with (22)-(23). References [10]-[12] have again supplied the motivation for this type of analysis.

We shall again assume that the coordinate functions $y_n(x)$ and $v_n(x)$ are those given in (13) and (34). Since (24)-(27) are still applicable, we need only derive an expression for $N_n(x)$ before proceeding to the numerical work. To this end, we find, after integrating the first equation in (46) and imposing the boundary conditions in (46),

$$N(x) = Q\, v(x) - \omega^2 \int_1^x (x-t)v(t)dt.$$

If we insert (32) into this last result, we obtain

$$N(x) = \sum_{n=1}^{N} b_n N_n(x), \tag{47}$$

where

$$N_n(x) = Q\, v_n(x) - \omega^2 \int_1^x (x-t)v_n(t)dt$$

$$= Q\, v_n(x) - \omega^2 h_n(x), \tag{48}$$

with

$$h_n(x) = \int_1^x (x-t)v_n(t)dt = \frac{2[Q^2+3(n^2+n-4)Q+2(n+1)(n+2)(n+3)(3n+10)]}{(n+3)(n+4)(n+5)} -$$

$$- \frac{2x[Q^2+3(n-1)(n+2)Q+6(n+1)(n+2)^2(n+3)]}{(n+2)(n+3)(n+4)} +$$

$$+ x^{n+3}\left(\frac{\alpha_n}{(n+2)(n+3)} - \frac{2\beta_n x}{(n+3)(n+4)} + \frac{\gamma_n x^2}{(n+4)(n+5)}\right).$$

Substitution of (26), (47), (10), and (32) into (44) yields

$$\delta \sum_{n=1}^{N} \sum_{k=1}^{N} a_n \, b_k \, J_{nk} \;=\; 0, \tag{49}$$

where

$$J_{nk} \equiv \int_0^1 [M_n(x)N_k(X) - Q\, y_n'(x)v_k'(x) - \omega^2 y_n(x)v_k(x)]\,dx + Q\, y_n'(1)v_k(1)$$

$$= \int_0^1 \{M_n(x)N_k(x) + [Q\, y_n''(x) - \omega^2 \cdot {}_n(x)]v_k(x)\}dx.$$

In view of (13), (27), (34), and (48), the coefficients J_{nk} may now be considered completely known. Proceeding with (49) as we did with (35), we find

$$\sum_{n=1}^{N} a_n \, J_{nk} \;=\; 0,$$

and consequently

$$\det\,(J_{nk}) \;=\; 0. \tag{50}$$

Numerical calculations with (50) yield, for $N = 2$,

$$Q_{cr} = 2.061\pi^2, \tag{51}$$

the most accurate of all the approximate calculations performed here.

To summarize the numerical results obtained by the various methods described above, the values of Q_{cr} are now tabulated in Table I along with the percentage of error relative to Beck's exact value.

TABLE I - COMPARISON OF THE VARIOUS METHODS

EQUATION NUMBER	Q_{cr}/π^2	% ERROR
(8) Exact	2.031	--
(16)	2.193	7.98
(28)	2.095	3.15
(39)	1.971	-2.96
(51)	2.061	1.48

Examination of Table I reveals that the Ritz procedure based upon the variational equation in (9) yields the poorest result, which is almost 8% greater than the exact value for Q_{cr}. Using the variant (21) of (9), i.e., the method of complementary energy, we obtain a much better approximation, namely (28), which is about 3% greater than the exact value. A slightly better approximation, (39), was obtained by means of the Ritz method based upon the adjoint variational principle in (31). However, the best approximation of all, (51), was obtained through a scheme which combined the methods of the adjoint variational principle and the complementary energy. In this later case, the approximate value of Q_{cr} was only about 1 1/2% greater than the exact value. It must be admitted that the improved value given in (51) was obtained at the expense of (i) introducing the adjoint problem (29) - (30) and the adjoint variational principle (31), (ii) selecting a set of coordinate functions for the adjoint problem (34), and (iii) deriving expressions for the "bending moments" $M_n(x)$, $N_n(x)$ by integrating (45) and (46). It might be argued from the standpoint of the labor involved that it is more efficient to use the ordinary Galerkin procedure (17) with $N \geq 3$.

HAUGER'S PROBLEMS

In the previous paragraphs we discussed Beck's problem for which it is possible to obtain the exact value of the critical load. Using certain methods of approximation, we calculated approximate values for Q_{cr} and compared these results with the known exact value. In many nonconservative stability problems it is a very difficult task to solve the differential equations of motion exactly, and consequently the value of the critical load parameter can be found only with considerable effort. This is particularly true for differential equations that have variable coefficients. In such cases the methods described in the preceding paragraphs offer rather efficacious tools for solving nonconservative stability problems. Hauger [29] considered problems of this type, namely, he computed approximately the critical loads for beams subjected to (i) uniformly distributed and (ii) linearly distributed tangential compressive loads.

We shall concern ourselves here with two problems studied by Hauger in which a linearly distributed tangential load is acting along the length of the beam. If the boundary conditions for the beam are either one end clamped - one end free or one end clamped - one end hinged, then the kinetic method must be used to determine the critical load because these systems become unstable through flutter. The equation of motion is

$$EI \frac{\partial^4 w}{\partial \bar{x}^4} + \frac{1}{2} q_0 (\ell - \bar{x})^2 \frac{\partial^2 w}{\partial \bar{x}^2} + \rho A \frac{\partial^2 w}{\partial \bar{t}^2} = 0. \tag{52}$$

Here q_o denotes the intensity of the tangential load and the remaining quantities are identical with those defined in (1). If we introduce the same length and time parameters that we used to derive (3), then (52) can be expressed as

$$\frac{\partial^4 w}{\partial x^4} + Q(1-x)^2 \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial t^2} = 0, \qquad o<x<1, \quad t>o, \tag{53}$$

where

$$Q = \frac{q_o \ell^4}{2EI}.$$

Setting $w(x,t) = y(x)\, e^{i\omega t}$, we reduce (53) to the following ordinary differential equation with variable coefficients:

$$y^{IV}(x) + Q(1-x)^2\, y''(x) - \omega^2 y(x) = 0, \qquad o<x<1. \tag{54}$$

For a clamped-free beam the boundary conditions are

$$y(o) = y'(o) = y''(1) = y'''(1) = 0 \tag{55}$$

and for a clamped-hinged beam they are

$$y(o) = y'(o) = y(1) = y''(1) = 0. \tag{56}$$

Henceforth, we shall refer to (54) subject to (55) as Hauger's first problem or Problem I and to (54) subject to (56) as Hauger's second problem or Problem II.

Because of the difficulty of solving these two boundary value problems exactly, Hauger resorted to Galerkin's procedure in its classical form, employing clamped-free and clamped-hinged beam modes

(obtained from (54) - (56) with $Q = 0$) for the coordinate functions. According to Leipholz investigations [4] - [7], this procedure should converge to the exact value of the critical load intensity Q_{cr} provided that a sufficient number of terms is taken in the approximation for $y(x)$. Hauger used only two-term approximations and calculated the following values for the critical load intensity parameter

$$Q_{cr} = 8.01\pi^2 \quad \text{and} \quad Q_{cr} = 20.38\pi^2, \tag{57}$$

for Problems I and II, respectively.

It is our intention here to reconsider these problems and to demonstrate by numerical calculation that these values are, respectively, 4.98% and 28.3% greater than the correct values. In view of the outcome of the calculations tabulated in Table I for Beck's problem, we shall elect to introduce the appropriate form of the adjoint variational principle for Hauger's problem. The numerical calculations to be presented here will be based on the methods of Galerkin and complementary energy in conjunction with the adjoint variational principle.

THE GALERKIN PROCEDURE AND THE ADJOINT VARIATIONAL PRINCIPLE

It was shown in [27] that the differential equation that is the adjoint of (54) is

$$v^{IV}(x) + Q\,[(1-x)^2 v(x)]'' - \omega^2 v(x) = 0, \quad o<x<1, \tag{58}$$

while the boundary conditions for the adjoint problem are identical to (55) and (56) for Problems I and II.

Furthermore, it was shown that these two boundary value problems may be generated from the variational principle

$$\delta\int_o^1\{y''(x)v''(x) - Q\, y'(x)[(1-x)^2v(x)]' - \omega^2 y(x)v(x)\}dx = 0. \tag{59}$$

We next assume that

$$y(x) = \tilde{y}(x) = \sum_{n=1}^{N} a_n y_n(x), \quad w(x) = \tilde{v}(x) = \sum_{n=1}^{N} b_n v_n(x), \tag{60}$$

where, for the present, the coordinate functions $y_n(x)$, $v_n(x)$ are assumed to satisfy only the geometric boundary conditions of a given problem. Thus, for Hauger's first problem, we assume that

$$y_n(o) = y_n'(o) = v_n(o) = v_n'(o) = 0, \tag{61}$$

whereas for Hauger's second problem

$$y_n(o) = y_n'(o) = y_n(1) = v_n(o) = v_n'(o) = v_n(1) = 0. \tag{62}$$

Substitution of (60) into (59) yields

$$\delta \sum_{n=1}^{N} \sum_{k=1}^{N} a_n b_k \int_o^1\{y_n''(x)v_k''(x) - Q\, y_n'(x)[(1-x)^2 v_k(x)]' -$$

$$- \omega^2 y_n(x)v_k(x)\}dx = 0. \tag{63}$$

Integrating by parts, we can easily show that

$$\int_0^1 y_n''(x)v_k''(x)dx = \begin{cases} \int_0^1 y_n^{IV}(x)v_k(x)dx + y_n''(1)v_k'(1) - y_n'''(1)v_k(1), & \text{(Problem I)} \\ \int_0^1 y_n^{IV}(x)v_k(x)dx + y_n''(1)v_k'(1), & \text{(Problem II)} \end{cases}$$

$$\int_0^1 y_n'(x)[(1-x)^2 v_k(x)]'dx = -\int_0^1 (1-x)^2 y_n''(x)v_k(x)dx,$$

where we have imposed the geometric boundary conditions (61) and (62). Inserting these last two results into (63), we obtain

$$\delta\sum_{n=1}^{N}\sum_{k=1}^{N} a_n b_k \{\int_0^1 L[y_n(x)]v_k(x)dx + y_n''(1)v_k'(1)-y_n'''(1)v_k(1)\} = 0 \tag{64}$$

and

$$\delta\sum_{n=1}^{N}\sum_{k=1}^{N} a_n b_k \{\int_0^1 L[y_n(x)]v_k(x)dx + y_n''(1)v_k'(1)\} = 0 \tag{65}$$

for Problems I and II, respectively, where

$$L[y_n(x)] \equiv y_n^{IV}(x) + Q(1-x)^2 y_n''(x) - \omega^2 y_n(x).$$

Proceeding with (64) and (65) in the usual fashion, we arrive at

$$\sum_{n=1}^{N} a_n \{\int_0^1 L[y_n(x)]v_k(x)dx + y_n''(1)v_k'(1) - y_n'''(1)v_k(1)\} = 0$$

and

$$\sum_{n=1}^{N} a_n \{\int_0^1 L[y_n(x)]v_k(x)dx + y_n''(1)v_k'(1)\} = 0$$

or equivalently

$$\int_0^1 L[\tilde{y}(x)]v_k(x)dx + \tilde{y}''(1)v_k'(1) - \tilde{y}'''(1)v_k(1) = 0, \tag{66}$$

and

$$\int_0^1 L[\tilde{y}(x)]v_k(x)dx + \tilde{y}''(1)v_k'(1) = 0 \tag{67}$$

for Problems I and II, respectively. Eqs. (66) and (67) are in the "semi-extended" form of Galerkin's procedure as described in [2] except that the weight functions, $v_k(x)$, here are the coordinate functions of the adjoint problem rather than of the original problem as is normally the case.

Because the speed of convergence is faster, in general, when the coordinate functions are selected so as to satisfy all the geometric and natural boundary conditions of a given problem, we shall elect to meet this requirement, thereby reducing (66) and (67) for both Problems I and II to

$$\int_0^1 L[\tilde{y}(x)]v_k(x)dx = 0. \tag{68}$$

Moreover, since the boundary conditions of the original and adjoint problems are identical, we shall select the y_n's and v_n's to be identical. Then (68) becomes

$$\int_0^1 L[\tilde{y}(x)]y_k(x)dx = 0, \tag{69}$$

which is the Galerkin procedure in its familiar form. Our numerical calculations will be based upon (69).

For Problem I, the clamped-free beam, the coordinate functions in (13) are again suitable. Numerical calculations were performed with (69) and (10) for $N = 2,3,\ldots,7$, and the results are shown in Table II.

TABLE II - THE CRITICAL LOAD INTENSITY

FOR HAUGER'S FIRST PROBLEM

N	Q_{cr}/π^2
2	7.5776
3	7.7830
4	7.6446
5	7.6310
6	7.6318
7	7.6314

For Problem II, the clamped-hinged beam, the set of polynomial coordinate functions

$$y_n(x) = (n+2)x^{n+1} - (2n+3)x^{n+2} + (n+1)x^{n+3}, \qquad n \geq 1 \tag{70}$$

satisfies all the geometric and natural boundary conditions in (56). Again using (69) and (10), we obtain the results exhibited in Table III.

TABLE III - THE CRITICAL LOAD INTENSITY

FOR HAUGER'S SECOND PROBLEM

N	Q_{cr}/π^2
2	19.698
3	16.525
4	16.315
5	15.914
6	15.886
7	15.885

THE COMPLEMENTARY ENERGY METHOD

Before commenting on the numerical results presented in Tables II and III in relation to Hauger's values in (57), we shall re-solve these problems using now the method of complementary energy. Introducing again (43), we may express (59) as

$$\delta\int_0^1 \{M(x)N(x)-Q\, y'(x)[(1-x)^2 v(x)]' - \omega^2 y(x)v(x)\}dx = 0, \tag{71}$$

with the auxiliary conditions

$$\begin{aligned} M''(x) &= Q(1-x)^2 y''(x) - \omega^2 y(x), \\ N''(x) &= Q[(1-x)^2 v(x)]'' - \omega^2 v(x), \end{aligned} \tag{72}$$

subject to

$$M(1) = M'(1) = N(1) = N'(1) = 0 \tag{73}$$

for Problem I and

$$M(1) = N(1) = 0 \tag{74}$$

for Problem II, which are consistent with (54) - (56), (58).

For Hauger's first problem we shall assume that

$$y(x) = \sum_{n=1}^{N} a_n y_n(x), \qquad v(x) = \sum_{n=1}^{N} b_n y_n(x), \tag{75}$$

where the $y_n(x)$'s, which satisfy all the boundary conditions in (55), are given in (13). By virtue of (72) and (73), we may also write

$$M(x) = \sum_{n=1}^{N} a_n M_n(x), \quad N(x) = \sum_{n=1}^{N} b_n N_n(x), \tag{76}$$

where

$$M_n''(x) = Q(1-x)^2 y_n''(x) - \omega^2 y_n(x), \qquad M_n(1) = M_n'(1) = 0,$$

$$N_n''(x) = Q[(1-x)^2 y_n(x)]'' - \omega^2 y_n(x), \; N_n(1) = N_n'(1) = 0. \tag{77}$$

Integration of (77) yields

$$M_n(x) = Q[(1-x)^2 y_n(x) + 2(x+2) r_n(x) - 6 s_n(x)] - \omega^2 g_n(x), \tag{78}$$

$$N_n(x) = Q(1-x)^2 y_n(x) - \omega^2 g_n(x), \tag{79}$$

where

$$r_n(x) = \int_1^x y_n(t)dt = -\frac{6}{n+4} + \frac{x^{n+2}}{2}\left(n+3 - 2nx + \frac{n(n+1)}{n+4} x^2 \right),$$

$$s_n(x) = \int_1^x t y_n(t)dt = -\frac{2(3n+10)}{(n+4)(n+5)} + \frac{x^{n+3}}{2}\left(n+2 - \frac{2n(n+3)}{n+4} x + \frac{n(n+1)}{n+5} x^2\right),$$

$$g_n(x) = x r_n(x) - s_n(x) = \frac{2(3n+10)}{(n+4)(n+5)} - \frac{6x}{n+4} + \frac{x^{n+3}}{2}\left(1 - \frac{2nx}{n+4} + \frac{n(n+1)x^2}{(n+4)(n+5)}\right).$$

Substituting (75) and (76) into (71), we find

$$\delta \sum_{n=1}^{N} \sum_{k=1}^{N} a_n b_k \, I_{nk} = 0,$$

where

$$I_{nk} = \int_0^1 [M_n(x) N_k(x) = Q(1-x)^2 y_n''(x) y_k(x) - \omega^2 y_n(x) y_k(x)] dx.$$

This leads to

$$\sum_{n=1}^{N} a_n \, I_{nk} = 0. \tag{80}$$

The numerical results obtained from (80) are shown in Table IV.

TABLE IV - THE CRITICAL LOAD INTENSITY FOR HAUGER'S FIRST PROBLEM

N	Q_{cr}/π^2
2	7.573
3	7.653
4	7.631
5	7.631

For Hauger's second problem we may again adopt (75), however now the coordinate functions will be those which appear in (70). Then with (70) and (75) integration of (72) subject to (74) leads to

$$M(x) = a_o(x-1) + \sum_{n=1}^{N} a_n M_n(x),$$
$$N(x) = b_o(x-1) + \sum_{n=1}^{N} b_n N_n(x), \tag{81}$$

where

$$a_o = M'(1), \qquad b_o = N'(1),$$

and $M_n(x)$ and $N_n(x)$ are given by (78) and (79), respectively, with

$$r_n(x) = \int_1^x y_n(t)dt = \frac{-3}{(n+3)(n+4)} + x^{n+2}\left(1 - \frac{(2n+3)}{n+3}x + \frac{(n+1)}{n+4}x^2\right),$$

$$s_n(x) = \int_1^x t\, y_n(t)dt = \frac{-(3n+7)}{(n+3)(n+4)(n+5)} + x^{n+3}\left(\frac{n+2}{n+3} - \frac{2n+3}{n+4}x + \frac{n+1}{n+5}x^2\right),$$

$$g_n(x) = x\, r_n(x) - s_n(x) = \frac{3n+7}{(n+3)(n+4)(n+5)} - \frac{3x}{(n+3)(n+4)} +$$

$$+ \quad x^{n+3}\left(\frac{1}{n+3} - \frac{(2n+3)x}{(n+3)(n+4)} + \frac{(n+1)x^2}{(n+4)(n+5)}\right).$$

Inserting (75) and (81) into the variational principle (71), we find

$$\delta\{\frac{1}{3}\, a_o b_o - a_o \sum_{n=1}^{N} b_n K_n - b_o \sum_{n=1}^{N} a_n J_n + \sum_{n=1}^{N} \sum_{k=1}^{N} a_n b_k\, I_{nk}\} = 0, \tag{82}$$

where

$$I_{nk} = \int_0^1 \{M_n(x) N_k(x) + Q(1-x)^2 y_n''(x) y_k(x) - \omega^2 y_n(x) y_k(x)\} dx,$$

$$J_n = \int_0^1 (1-x) M_n(x) dx, \qquad K_n = \int_0^1 (1-x) N_n(x) dx.$$

But (82) leads to

$$\frac{1}{3}\, a_o - \sum_{n=1}^{N} a_n J_n \quad \delta b_o + \sum_{k=1}^{N} \quad \sum_{n=1}^{N} a_n\, I_{nk} - a_o K_k \quad \delta b_k +$$

$$+ \quad \frac{1}{3}\, b_o - \sum_{n=1}^{N} b_n K_n \quad \delta a_o + \sum_{n=1}^{N} \quad \sum_{k=1}^{N} b_k\, I_{nk} - b_o J_n \quad \delta a_n = 0. \tag{83}$$

In the usual fashion, we obtain from (83)

$$a_o - 3 \sum_{n=1}^{N} a_n J_n = 0, \qquad \sum_{n=1}^{N} a_n\, I_{nk} - a_o K_k = 0, \tag{84}$$

plus two additional equations which can easily be shown to be equivalent to (84). Eliminating a_o between the two equations in (84), we arrive at

$$\sum_{n=1}^{N} a_n \left[I_{nk} - 3\, J_n\, K_k \right] = 0. \tag{85}$$

The numerical calculations obtained from (85) are shown in Table V.

TABLE V - THE CRITICAL LOAD INTENSITY FOR HAUGER'S SECOND PROBLEM

N	Q_{cr}/π^2
2	16.522
3	16.012
4	15.929
5	15.884

Therefore, in view of Tables II - V, we conclude that accurate values for the critical load intensity parameter, Q_{cr}, are

$$Q_{cr} = 7.631\pi^2 \quad \text{and} \quad Q_{cr} = 15.88\pi^2$$

for Hauger's first and second problems, respectively. These results are 4.98% and 28.3% lower than the corresponding numerical values reported by Hauger (see Eq. 57) who used a two-term Galerkin approximation with clamped-free and clamped-hinged beam modes as coordinate functions. Here for coordinate functions we have used polynomials that satisfy all the geometric and natural boundary conditions of the respective problems in (i) the Galerkin procedure and (ii) the method

of complementary energy in conjunction with the so-called adjoint variational principle to calculate approximate values for Q_{cr}. The application of these processes was continued by successively increasing the number of terms included in the approximation functions until the numerical values appeared to converge to the correct value for Q_{cr}. In both cases it appears that the method of complementary energy converged at a somewhat faster rate than did the Galerkin procedure. The method of complementary energy requires extra effort to apply relative to the Galerkin method since the "bending moments" must be obtained by integrating an elementary second order linear differential equation. The Galerkin procedure offers the advantage that the numerical calculations may be set up in a rather straightforward fashion. At least in the numerical computations reported here, the application of the Galerkin procedure offered no computational hardships as the value of the integer N increased because all the integrals involved were evaluated very efficiently on the computer by means of a double precision subroutine based upon the Gaussian quadrature technique which evaluates integrals of polynomials exactly. Another subroutine was used to evaluate the various determinants that arose, and a root finding scheme determined the value of the load intensity parameter for a given value of the frequency of vibration of the system.

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